Assignment 10.

1. Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (\cos x \cos y)^2,$$

obtaining an expression for $\tan y$ in terms of x. [6]

$$\int \frac{dy}{\cos^2 y} = \int \cos^2 x \, dx$$

$$\tan y = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

2. Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \ln(x^y),$$

obtaining an expression for y in terms of x. [5]
Given further that $y = 1$ when $x = \mathrm{e}$, find the value of y when $x = 1$. [2]

$$\int \frac{dy}{y} = \int \ln x \, dx$$

$$\ln y = x\ln x - x + C$$

$$y = e^{x\ln x - x + C} \Rightarrow y(1) = e^{0-1} = e^{-1}$$

$$1 = e^{e - e + C} \Rightarrow C = 0$$

3. Given that the curve, whose equation satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x\sqrt{(x^2+1)(y+1)},$$

passes through the point $(1, 1)$, find an expression of y in terms of x. [7]

$$\int \frac{dy}{\sqrt{y+1}} = \int 3x\sqrt{1+x^2}\, dx$$

$$2(1+y)^{\frac{1}{2}} = \frac{3}{2} \cdot \frac{2}{3}(1+x^2)^{\frac{3}{2}} + C$$

$$\Rightarrow \sqrt{1+y} = \frac{(1+x^2)^{\frac{3}{2}}}{2} + C'$$

$$\sqrt{2} = \frac{2\cdot\sqrt{2}}{2} + C \Rightarrow C = 0$$

$$\Rightarrow y = \frac{(1+x^2)^3}{4} - 1$$

4. In ecology, a common model of population growth was proposed by *Pierre-François Verhulst*, where the rate of reproduction is proportional to both the existing population and the amount of available resources, *ceteris paribus* (all else being equal). The model is formalized by the differential equation:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP \cdot \left(1 - \frac{P}{K}\right),$$

where P represents population size, t represents time, and r, K are two positive constants.

(a) Given the initial condition: $P = P_0$, when $t = 0$, solve the differential equation and express P in terms of t, r, K and P_0. [7]

(b) According to *Verhulst*'s model, what is the limiting population size in the long run? [1]

$$\int \frac{dP}{P(1-\frac{P}{K})} = \int r\,dt \qquad \ln \frac{P_0}{K-P_0} = C$$

$$\frac{1}{P} + \frac{1}{K-P}$$

$$K \int \frac{dP}{P(K-P)} = \int r\,dt \qquad \Rightarrow \frac{\frac{P}{K-P}}{\frac{P_0}{K-P_0}} = e^{rt}$$

$$\ln P - \ln(K-P) = rt + c$$

$$\Rightarrow P = \frac{K P_0}{K-P_0} e^{rt} \Big/ 1 + \frac{P_0}{K-P_0}$$

$t \to \infty, \ P \to K$

5. A tank is being filled with water. At time t minutes after filling begins, the volume of water is V liters. Water is poured in at a constant rate of 9 liters per minute, but owing to leakage, it is lost at a rate proportional to V. Initially the tank is empty. When $V = 4$, $\frac{\mathrm{d}V}{\mathrm{d}t} = 7$.

(a) Show that V satisfies the differential equation: $\frac{\mathrm{d}V}{\mathrm{d}t} = 9 - \frac{1}{2}V$. [2]

(b) Solve the above differential equation, expressing V in terms of t. [4]

(c) Calculate the time taken to fill the tank with 9 liters of water. [2]

$9 - 4K = 7 \Rightarrow K = \frac{1}{2}$

(a) $\frac{dV}{dt} = 9 - KV$

$-2\ln(9 - \frac{1}{2}V) = t + c$

(b) $\int \frac{dV}{9 - \frac{1}{2}V} = \int dt \Rightarrow$

$\Rightarrow 9 - \frac{1}{2}V = e^{\frac{t+c}{-2}}$

$\Rightarrow t = 0, V = 0 \Rightarrow e^{\frac{C}{2}} = 9$ $C = \cancel{18}$

6. (†) Solve the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 3y^2,$$

such that $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 4$ when $x = 1$. [7]

Hint: Prove that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = z\frac{\mathrm{d}z}{\mathrm{d}y}$, where $z = \frac{\mathrm{d}y}{\mathrm{d}x}$.

$9 - \frac{1}{2}V = 9 \cdot e^{-\frac{t}{2}}$

$\Rightarrow V = 18 - 18e^{-\frac{t}{2}}$

Total mark of this assignment: $36 + 7$.

$9 = 18 - 18e^{-\frac{t}{2}}$

The symbol (†) indicates a bonus question. Finish other questions before working on this one.

$e^{-\frac{t}{2}} = \frac{1}{2} \Rightarrow t = 2\ln 2$